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MAXIMAL QUOTIENTS OF SEMIPRIME PI-ALGEBRAS

BY

LOUIS HALLE ROWEN[1]

ABSTRACT. J. Fisher [3] initiated the study of maximal quotient rings of semiprime PI-rings by noting that the singular ideal of any semiprime PI-ring R is 0; hence there is a von Neumann regular maximal quotient ring $Q(R)$ of R. In this paper we characterize $Q(R)$ in terms of essential ideals of $C = \text{cent } R$. This permits immediate reduction of many facets of $Q(R)$ to the commutative case, yielding some new results and some rapid proofs of known results. Direct product decompositions of $Q(R)$ are given, and $Q(R)$ turns out to have an involution when R has an involution.

1. Introduction. Let Ω be a commutative algebra with 1 and let R be a semiprime Ω-algebra, not necessarily with 1 (by *semiprime* we mean R has no nonzero nilpotent ideals, where all ideals are understood to be Ω-invariant; equivalently, the intersection of the prime ideals of R is 0). Let the *standard polynomial on k letters* $S_k(X_1, \cdots, X_k) \equiv \Sigma_\pi (\text{sg } \pi) X_{\pi 1} \cdots X_{\pi k}$, π ranging over the permutations of $(1, \cdots, k)$; a polynomial $f(X_1, \cdots, X_m)$ (with coefficients in Ω) is an *identity* of R if, evaluated in R, $f(X_1, \cdots, X_m) = 0$, each $r_1, \cdots, r_m$ in R. The semiprime algebra R is a PI*-algebra of degree n* if S_{2n} is an identity of R but S_{2n-2} is not an identity of R. Throughout this paper we assume R is a semiprime PI-algebra of finite degree n, and we let $C = \text{cent } R$. Formanek [4] has shown there exists a polynomial $g(X_1, \cdots, X_{n+1})$ with integral coefficients, one of which is ± 1, such that each of $X_2, \cdots, X_{n+1}$ has degree 1 in each monomial of g; moreover, evaluated in R, $g(r_1, \cdots, r_{n+1}) \in C$ for each $r_1, \cdots, r_{n+1}$ in R, and there exist $r_1, \cdots, r_{n+1}$ in R such that $g(r_1, \cdots, r_{n+1}) \neq 0$ (in particular $C \neq 0$). An application of Formanek's polynomials is

Theorem A (Rowen [8]). *If A is a nonzero ideal of R then $A \cap C \neq 0$.*

Now for subsets V, W of R, define $\text{Ann}_V W = \{v \in V | Wv = 0\}$ and $\text{Ann}'_V W =$

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$\{v \in V | vW = 0\}$. If $V = R$ then the subscript will be omitted. Clearly, for any ideal A of R, $\mathrm{Ann}\, A = \mathrm{Ann}'\, A$ (*Proof.* $(A\, \mathrm{Ann}'\, A)^2 = A((\mathrm{Ann}'\, A)A)\mathrm{Ann}'\, A = 0$, so $A\, \mathrm{Ann}'\, A = 0$ since R is semiprime. Hence $\mathrm{Ann}'\, A \subseteq \mathrm{Ann}\, A$ and, symmetrically, $\mathrm{Ann}\, A \subseteq \mathrm{Ann}'\, A$.) Similarly, one sees that for any ideals A, B of R, $AB = 0 \Leftrightarrow A \cap B = 0$.

Call a (left, right, 2-sided) ideal J of R (*left*, *right*, 2-*sided*) *essential* if $J \cap B \neq 0$ for all nonzero (left, right, 2-sided) ideal of R. (The word "2-sided" will often be omitted for convenience.) An ideal J of R is essential if and only if $\mathrm{Ann}\, J = 0$, if and only if $\mathrm{Ann}'\, J = 0$ (by the preceding paragraph); we conclude each essential ideal of R is left essential and right essential. (Indeed, suppose J is an essential ideal of R and B is a left ideal of R such that $J \cap B = 0$. Then $JB \subseteq J \cap B = 0$, so $B \subseteq \mathrm{Ann}\, J = 0$.) By a *left essential ideal* we mean a left essential left ideal. If J is a left essential ideal of R then $\mathrm{Ann}'\, J = 0$. Let $Z = \{r \in R |$ there exists a left essential ideal J of R such that $r \in \mathrm{Ann}\, J\}$. Z is well known to be an ideal of R, called the *left singular ideal*. The *right singular ideal* Z' is defined analogously.

Proposition 1 (Fisher [3]). $Z = Z' = 0$.

Proof (Martindale [6]). Let $c \in Z \cap C$. Then $c \in \mathrm{Ann}'\, J$ for some left essential ideal of R. But $\mathrm{Ann}'\, J = 0$, so $Z \cap C = 0$. Therefore, $Z = 0$ by Theorem A. Likewise $Z' = 0$. Q.E.D.

In this case, it is well known (cf. Johnson [5]) that the left injective hull of R has a natural ring structure $Q(R)$. $Q(R)$ can be characterized in terms of essential ideals, as follows (cf. Martindale [6]):

(a) There is a canonical injection $R \hookrightarrow Q(R)$ by which we view $R \subseteq Q(R)$.

(b) For any left essential ideal J of R and for any f in $\mathrm{Hom}_R(J, R)$ (as left R-modules), there exists q in $Q(R)$ such that $xq = f(x)$, all x in J.

(c) For any given q in $Q(R)$ there is a left essential ideal J of R such that $Jq \subseteq R$.

(d) $q = 0$ if and only if $Jq = 0$ for some left essential ideal J of R.

There is a natural way to extend the algebra structure of R to $Q(R)$. Namely, given q in $Q(R)$, ω in Ω, let J be a left essential ideal of R such that $Jq \subseteq R$, and define f in $\mathrm{Hom}_R(J, R)$ by $f(x) = \omega(xq)$, all x in J. By (b), we may pick q_1 in $Q(R)$ such that $xq_1 = f(x)$, all x in J; define ωq to be q_1. To see that ωq is well defined, suppose J' is another left essential ideal of R such that $J'q \subseteq R$, and define f' in $\mathrm{Hom}_R(J', R)$ by $f'(x) = \omega(xq)$, all x in J'; let q_1' in Q be such that $xq_1' = f'(x)$, all x in J'. For all x in $J \cap J'$, $xq_1 = f(x) = \omega(xq) = f'(x) = xq_1'$, so $(J \cap J')(q_1 - q_1') = 0$. Since $J \cap J'$ is left essential, $q_1 = q_1'$ by (d); hence ωq is well defined, extending the algebra structure on R. Similar verifications show that $Q(R)$ is now an algebra, called henceforth the *maximal left quotient* *algebra of* R; for any $f \in \operatorname{Hom}(J, R)$, J left essential, we have $f(\omega x) = \omega f(x)$ for all ω in Ω, x in J.

2. A central characterization of $Q(R)$.

Lemma 1 (Martindale [6]). *Any left essential ideal J of R is itself a semiprime PI-algebra, and* cent $J = J \cap C$.

Proof. Straightforward application of Proposition 1. The following lemma is also known by Martindale [6], but a different proof is used to avoid reliance on the other results in [6].

Lemma 2.(i) *If J is a left essential ideal of R then $J \cap C$ intersects nontrivially all ideals of R. Hence $(J \cap C)R$ is 2-sided essential.*

(ii) *A left ideal J of R is left essential if and only if $(J \cap C)$ is essential in C.*

Proof. (i) Suppose B is an ideal of R such that $(J \cap C) \cap B = 0$. Then $(J \cap C) \cap (J \cap B) = 0$, so by Theorem A applied to the semiprime PI-algebra J (with center $J \cap C$), we conclude $J \cap B = 0$. Hence $B = 0$, so (i) follows immediately.

(ii) Suppose J is left essential and let $B = \operatorname{Ann}_C(J \cap C)$. Clearly $BR(J \cap C) = 0$, so $(BR \cap (J \cap C))^2 = 0$; hence $BR \cap (J \cap C) = 0$, implying $BR = 0$ from (i). Therefore $B = 0$, so $J \cap C$ is essential in C.

Conversely, suppose $(J \cap C)$ is essential in C and let B be a left ideal of R such that $J \cap B = 0$. Then $(BR \cap C)(J \cap C) \subseteq B(J \cap C)R \subseteq (B \cap J)R = 0$, so $BR \cap C = 0$. Therefore $BR = 0$, by Theorem A, so $B = 0$ and J is left essential. Q.E.D.

The routine preliminaries have been set for the main theorem:

Theorem 1. *$Q(R)$ is characterized by the following properties*:

(i) *There is a canonical injection $R \hookrightarrow Q(R)$ sending C into* cent $Q(R)$.

(ii) *For any essential ideal E of C and for any f in* $\operatorname{Hom}_C(E, R)$, *one can find q in $Q(R)$ such that $xq = f(x)$, all x in E.*

(iii) *For any q in $Q(R)$, $Eq \subseteq R$ for some essential ideal E of C.*

(iv) *$q = 0$ if and only if $Eq = 0$ for some essential ideal E of C.*

Proof. First we show (a)–(d) of the previous characterization imply (i)–(iv).

(i) $R \subseteq Q(R)$; we claim $C \subseteq$ cent $Q(R)$. Choose c in C, q in $Q(R)$. By (c), $Jq \subseteq R$ for some left essential ideal J of R, so, for all x in J, $0 = (xq)c - c(xq) = xqc - (cx)q = xqc = x(qc - cq)$, implying $qc - cq = 0$ by (d). Hence $C \subseteq$ cent $Q(R)$.

(ii) Let E be an essential ideal of C. Then $C \cap RE$ is surely essential in C, so, by Lemma 2, RE is essential in R. Given f in $\operatorname{Hom}_C(E, R)$ we wish to define $f'\colon RE \to R$ by $f'(\Sigma\, r_i c_i) = \Sigma\, r_i f(c_i)$, all c_i in E, all r_i in R. To check that f' is well defined, let $B = \{\Sigma\, r_i f(c_i) | \Sigma\, r_i c_i = 0,\ r_i$ in R, c_i in $E\}$, an ideal of R. If $B \neq 0$ then $B \cap C \neq 0$ by Theorem A, so $B \cap C \cap E \neq 0$ and we could choose nonzero $b = \Sigma\, r_i f(c_i)$ in $B \cap E \cap C$, with $\Sigma\, r_i c_i = 0$. But then $b^2 = b\, \Sigma\, r_i f(c_i) = \Sigma\, r_i f(bc_i) = \Sigma\, r_i f(c_i b) = (\Sigma\, r_i c_i) f(b) = 0$, contrary to C being semiprime. Hence $B = 0$ and f' is a well-defined element of $\mathrm{Hom}_R(RE, R)$. By (b), there exists q in $Q(R)$ such that $xq = f'(x)$ for all x in RE, implying (by (d)) $xq = f'(x)$ for all x in E.

(iii) By (c), $Jq \subseteq R$ for some left essential ideal J of R. Let $E = J \cap C$, an essential ideal in C by Lemma 2.

(iv) Immediate from (d) and Lemma 2.

Thus, the left maximal quotient algebra $Q(R)$ satisfies (i)–(iv). Conversely, assume some algebra Q satisfies (i)–(iv). We shall show $Q = Q(R)$ by verifying (a)–(d).

(a) Immediate from (i).

(b) Suppose J is left essential in R and $f \in \mathrm{Hom}_R(J, R)$. Then, by Lemma 2, $E = J \cap C$ is essential in C and surely $f \in \mathrm{Hom}_C(E, R)$. By (ii), there exists q in Q such that $f(c) = cq$, all c in E. For all x in J, all c in E, $c(f(x) - xq) = f(cx) - cxq = f(xc) - xcq = x(f(c) - cq) = 0$, so $E(f(x) - xq) = 0$, all x in J. Hence by (iv), $f(x) = xq$, all x in J.

(c) Immediate from (iii) and Lemma 2.

(d) Immediate from (iv) and Lemma 2. Q.E.D.

Corollary 1 (Martindale [6, Theorem 5]). *$Q(R)$ is also the right maximal quotient algebra of R.*

Proof. Conditions (i)–(iv) are left-right symmetric. Q.E.D.

Martindale also has shown $Q(R)$ satisfies all multilinear identities of R. Call a polynomial $f(X_1, \cdots, X_m)$ *homogeneous* if each monomial of f has the same total degree.

Corollary 2. *Each homogeneous identity of R is an identity of $Q(R)$.*

Proof. Let $f(X_1, \cdots, X_m)$ be a homogeneous identity of R, of total degree d. We wish to show, given any $q_1, \cdots, q_m$ in $Q(R)$, that $f(q_1, \cdots, q_m) = 0$. By Theorem 1 (iii) there are essential ideals E_i of C such that $E_i q_i \subseteq R$, $1 \leq i \leq m$. Let $E = E_1 \cap \cdots \cap E_m$, an essential ideal of C. For each c in E, $0 = f(cq_1, \cdots, cq_m) = c^d f(q_1, \cdots, q_m)$, so $0 = \hat{E} f(q_1, \cdots, q_m)$, where $\hat{E}$ is the ideal of C generated by $\{c^d | c \in E\}$. But $\hat{E}$ is essential in C; indeed, for any nonzero ideal B of C we can pick $b \neq 0$ in $B \cap E$, and then $0 \neq b^d \in B \cap \hat{E}$. Hence $f(q_1, \cdots, q_m) = 0$ by Theorem 1 (iv). Q.E.D.

Corollary 3 (Armendariz-Steinberg [2]). cent $Q(R) = Q(C)$.

Proof. Let $C' = \text{cent } Q(R)$. We need to verify (i)′–(iv)′, obtained from conditions (i)–(iv) of Theorem 1 by replacing R by C.

(i)′ $C \hookrightarrow C'$ is part of (i).

(ii)′ Given E essential in C and f in $\text{Hom}_C(E, C)$, Theorem 1 (ii) provides q in $Q(R)$ such that $xq = f(x)$, all x in E. It suffices to show $q \in C'$. Note $Eq \subseteq C \subseteq C'$; for all c in E, all q' in $Q(R)$, $0 = (cq)q' - q'(cq) = c(qq' - q'q)$, so $qq' - q'q = 0$ (by (iv)), all q' in $Q(R)$, implying $q \in C'$.

(iii)′, (iv)′ are immediate respectively from (iii), (iv). Q.E.D.

Incidentally, if R is the infinite direct sum $\bigoplus M_n(Q)$ then $Q(R)$ is the infinite direct product $\Pi\, M_n(Q)$, so $Q(C)R \neq Q(R)$ in this case (example due, I believe, to R. Snider).

Proposition 2. *For q in $Q(R)$, $q \in \text{cent } Q(R)$ if and only if there is a left essential ideal J of R such that $qx - xq = 0$, all x in J.*

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) Pick q' arbitrarily from $Q(R)$ and let E be an essential ideal of C such that $Eq' \subseteq R$ (cf. Theorem 1 (iii)). Then $E' = (J \cap C)E$ is essential in C and, for all c in E', we have $cq' \in J$ and $c(qq' - q'q) = q(cq') - (cq')q = 0$. Hence $qq' - q'q = 0$, all q' in $Q(R)$, implying $q \in \text{cent } Q(R)$. Q.E.D.

Theorem 2. *Let J be a 2-sided essential ideal of R. For any bimodule homomorphism $f: J \longrightarrow R$ there exists q in $\text{cent } Q(R)$ such that $f(r) = rq$, all r in J.*

Proof. Since f is a left module homomorphism and J is left essential, there exists q in $Q(R)$ such that $f(r) = rq$, all r in J. For all x, r in J, $xrq = xf(r) = f(xr) = f(x)r = xqr$, so $x(rq - qr) = 0$, implying $rq - qr = 0$, all r in J. By Proposition 2 $q \in \text{cent } Q(R)$. Q.E.D.

Remark. One could parallel the proof of Theorem 1 to show $Q(R)$ is actually the $Q_0(R)$ of Amitsur [1]. Hence, [1, Theorem 3] implies Proposition 2 and Theorem 2. Similarly, [1, Theorem 5] yields a nice proof that $Q(R)$ is von Neumann regular, a fact observed in general (for rings with zero left singular ideal) by Johnson [5].

3. **Structure of $Q(R)$.** In this section we assume $1 \in R$ and give two direct sum decompositions of $Q(R)$. (Note 1 is also the multiplicative unit of $Q(R)$.) The point of departure is the easily verified

Theorem B. *Viewing a left essential ideal J of R as a semiprime PI-algebra (by Lemma 1), we have $Q(J) \approx Q(R)$.*

Corollary 4. *If A, A' are ideals of R such that $A' = \text{Ann } A$ and $A = \text{Ann } A'$, then $Q(R) \approx Q(R/A) \oplus Q(R/A')$.*

Proof. Given in [9, Theorem 4]. Like Theorem B, one does not need the assumption R is a PI-ring but only requires that R has zero left singular ideal.

Say a prime ideal P of R has *degree* j if R/P has degree j. Let N_n be the intersection of those prime ideals of R with degree n, and, for $j < n$, let $N_j = \bigcap\{P$ prime in R of degree $j \mid P \not\supseteq N_i$, all $i > j\}$. Clearly, $N_1 \cap \cdots \cap N_n = 0$.

Theorem 3. $Q(R) \approx Q(R/N_1) \oplus \cdots \oplus Q(R/N_n)$.

Proof. First we show $Q(R) \approx Q(R/N_n') \oplus Q(R/N_n)$, where $N_n' = \bigcap_{i=1}^{n-1} N_i$. In view of Corollary 4, it suffices to show $N_n' = \operatorname{Ann} N_n$ and $N_n = \operatorname{Ann} N_n'$. Since $N_n N_n' = 0$, $N_n' \subseteq \operatorname{Ann} N_n$. On the other hand, it is easy to see $N_n' = \bigcap\{P \mid P \not\supseteq N_n\}$. For $P \not\supseteq N_n$, however, $\operatorname{Ann} N_n \subseteq P$ since $N_n \operatorname{Ann} N_n = 0 \subseteq P$; hence $\operatorname{Ann} N_n \subseteq N_n'$. Analogously, it is clear $N_n \subseteq \operatorname{Ann} N_n'$. Since R/N_n' has degree $\leq n-1$, $P \not\supseteq N_n'$ for each prime P of degree n, so, arguing as above, we have $\operatorname{Ann} N_n' \subseteq \bigcap\{P$ prime of degree $n\} = N_n$.

So $Q(R) \approx Q(R/N_n') \oplus Q(R/N_n)$. Since R/N_n' has degree $\leq n-1$, the theorem follows by induction on n. Q.E.D.

Armendariz-Steinberg [2] proved $Q(R)$ is a finite direct sum of Azumaya algebras of finite rank; we are now in a position to develop a straightforward proof of this fact, displaying at the same time the structure involved.

Let $g(X_1, \cdots, X_{n+1})$ be the Formanek polynomial described in §1; g happens to be homogeneous of total degree n^2 (cf. [4]). Let $I_g(R) = \{g(r_1, \cdots, r_{n+1}) \mid$ all $r_1, \cdots, r_{n+1}$ in $R\}$; note that $cg(r_1, \cdots, r_{n+1}) = g(r_1, \cdots, r_{n+1}c)$ for all c in C, so $I_g(R)$ is a monoid ideal of the (multiplicative) monoid C. Let $I_g'(R)$ be the additive subgroup generated by $I_g(R)$; $I_g'(R)$ is an ideal in C, and the prime ideals of R containing $I_g(R)$ are precisely those primes of degree $\leq n-1$. Also observe for any central idempotent e, eR is a semiprime PI-algebra of degree $\leq n$, with multiplicative unit e.

Definition. R is *stable* if $1 \in I_g(R)$.

Lemma 3. *If $e \in I_g(R)$ and e is a nonzero idempotent then eR is stable of degree n; i.e. $e \in I_g(eR)$.*

Proof. Let $e = g(r_1, \cdots, r_{n+1})$. Then $g(er_1, \cdots, er_{n+1}) = e^{n^2} g(r_1, \cdots, r_{n+1}) = ee = e$. Q.E.D.

Theorem 4. (i) *If every nonzero ideal of C contains a nonzero idempotent of $I_g(R)$ then $Q(R)$ is stable.*

(ii) *If $I_g'(R)$ is essential in C then $Q(R)$ is stable.*

Proof. (i) Using Zorn's lemma we find a collection of idempotents e_λ in $I_g(R)$ such that $\bigoplus e_\lambda R$ is an essential ideal of R. Then $Q(R) \approx Q(\bigoplus_\lambda e_\lambda R) \approx \prod_\lambda Q(e_\lambda R)$ canonically, so $\prod_\lambda e_\lambda = 1$ in $Q(R)$. But, by Lemma 3, $e_\lambda \in I_g(e_\lambda R) \subseteq I_g(Q(e_\lambda R))$, so $\prod_\lambda e_\lambda \in I_g(\prod Q(e_\lambda R))$. Hence $Q(R)$ is stable.

(ii) If $I_g'(R)$ is essential in C, then Theorem 1 (iii) implies $I_g'(Q(R))$ is essential in cent $Q(R)$. Since $Q(R) = Q(Q(R))$, we may replace R by $Q(R)$ and assume C is von Neumann regular (in view of Corollary 3). We shall conclude the proof of (ii) by showing that the hypothesis of part (i) is now satisfied. Indeed, let A be a nonzero ideal of C. Choose $a \neq 0$ in A and $r_1, \cdots, r_{n+1}$ in R such that $ag(r_1, \cdots, r_{n+1}) \neq 0$ (possible since $0 = \operatorname{Ann} I_g'(R) = \operatorname{Ann} I_g(R)$). Let $a' = ag(r_1, \cdots, r_{n+1}) = g(r_1, \cdots, ar_{n+1}) \in A \cap I_g$. Since C is von Neumann regular, there exists d in C such that $a'da' = a'$. But $a'd$ is a nonzero idempotent of $A \cap I_g$, as desired. Q.E.D.

Theorems 3 and 4 combine to show $Q(R)$ is always a direct sum of the stable semiprime PI-algebras $Q(R_1), \cdots, Q(R_n)$. But every stable semiprime PI-algebra is Azumaya of finite rank, by the celebrated Artin-Procesi theorem (cf. [7]), so we get Armendariz-Steinberg's result with an explicit construction.

We turn now to the question of whether $Q(R)$ can be decomposed into a direct product of simple artinian factors. First observe if R is prime then a simple extension of R, easily seen to be $Q(R)$, is obtained merely by inverting elements of C (cf. [8]). Conversely, there is

Theorem 5. *Let a semiprime Ω-algebra T be essential as a left R-module extension of R. Then for any inessential prime ideal P of T, $P \cap R$ is an inessential prime ideal in R.*

Proof. $0 \neq R \cap \operatorname{Ann}_T P \subseteq \operatorname{Ann}_R (P \cap R)$, so $P \cap R$ is inessential in R. To see $P \cap R$ is prime in R, let A, A' be ideals of R such that $AA' \subseteq P$, and pick q arbitrarily in T. Since $Q(R)$ is the maximal left essential extension of R, $T \subseteq Q(R)$ as left R-modules, so (by Theorem 1 (iii)) there is an essential ideal E of C such that $Eq \subseteq R$. Let $B = \operatorname{Ann}_R (P \cap R)$. $EBAqA' = BAEqA' \subseteq BAA' = 0$, so $BAqA' = 0$ by Theorem 1 (iv). But $B \not\subseteq P$ since T is semiprime, so $AqA' \subseteq P$, all q in T. Hence $ATA' \subseteq P$, implying $A \subseteq R \cap P$ or $A' \subseteq R \cap P$. Q.E.D.

Incidentally, it is well known and very easily seen that the module injection $T \hookrightarrow Q(R)$ is in fact a ring injection.

Corollary 5. *Let T be as in the theorem. If T is prime then R is prime (and thus $Q(R)$ is simple artinian).*

Proof. 0 is an inessential ideal of T, so 0 is prime in R. Q.E.D.

Call a ring *prime-essential* if all its primes are essential. Prime-essential semiprime PI-algebras exist, as shown in [9].

Corollary 6. *If R is prime-essential then T is prime-essential.*

Proof. Immediate from the theorem.

In [9], under the assumption R has zero left singular ideal (not necessarily a PI-algebra), $Q(R)$ is given canonically as the complete direct product of maximal left quotients of prime images and the maximal left quotient of a prime-essential ring, the latter being 0 if and only if $\bigcap\{P \mid P$ inessential prime ideal of $R\} = 0$. Hence, in view of Theorem 5, we have immediately

Theorem 6. *$Q(R)$ is the direct product of simple algebras and a prime-essential algebra. There is a direct summand, simple as an algebra, of $Q(R)$ if and only if R has an inessential prime ideal. There is an algebra $T \supseteq R$, essential as a left R-module and a product of simple algebras, if and only if the intersection of the inessential primes of R is 0; in this case we can take $T = Q(R)$.*

4. Maximal quotients of semiprime PI-algebras with involution. A *semiprime PI-algebra with involution* $(R, *)$ is a semiprime PI-algebra with antiautomorphism $(*)$ of degree ≤ 2. Note $(*)$ is an automorphism of degree ≤ 2 on C. Define $\operatorname{cent}(R, *) = \{c \in C \mid c^* = c\}$, and let $\hat{C} = \operatorname{cent}(R, *)$. If $\hat{C} = C$ then $(*)$ is of *first kind* on R; otherwise $(*)$ is of *second kind* on R.

Theorem 7. *If $(R, *)$ is a semiprime PI-algebra with involution then $Q(R)$ has an involution of the same kind as $(*)$, coinciding with $(*)$ on R.*

Proof. We use the characterization of $Q(R)$ in Theorem 1. Given q in $Q(R)$, choose an essential ideal E of C such that $Eq \subseteq R$. It is easy to show $E^* = \{c \in C \mid c^* \in E\}$ is an essential ideal of C; define $f: E^* \to R$ by $f(x) = (x^*q)^*$ for each x in E^*. For any c in C, $f(cx) = (x^*c^*q)^* = (c^*x^*q)^* = (x^*q)^*c = f(x)c = cf(x)$, so $f \in \operatorname{Hom}_C(E^*, R)$. Hence there is an element of $Q(R)$, which we shall call q^*, such that $xq^* = f(x)$ for all x in E^*. A straightforward verification shows q^* is independent of the choice of E, and $q \to q^*$ is an involution, coinciding with the given involution on R. In particular, if $(*)$ is of the second kind on R then $\operatorname{cent}(Q(R), *) \neq \operatorname{cent} Q(R)$, so $(*)$ is of the second kind on $Q(R)$.

On the other hand, suppose $(*)$ is of the first kind on R. Then, with notation as above, $E^* = E$ and f is given by $f(x) = (xq)^*$ for each x in E^*. If $q \in \operatorname{cent} Q(R)$ then $xq \in R \cap \operatorname{cent} Q(R) = C$, so $f(x) = xq$; therefore $q^* = q$ and $(*)$ is of the first kind on $Q(R)$.

An *ideal* of $(R, *)$ is an ideal of R, stable under $(*)$; an ideal of $(R, *)$ is *essential* if it intersects nontrivially each nonzero ideal of $(R, *)$.

Lemma 3. (i) *If A is an ideal of $(R, *)$ then* $\operatorname{Ann} A$ *is an ideal of* $(R, *)$.
(ii) *If J is an essential ideal in $(R, *)$ then J is essential in R.*
(iii) *If J is an essential ideal of R then JJ^* is essential in $(R, *)$.*

Proof. (i) Let $B = \operatorname{Ann} A$. $B^*A = (A^*B)^* = (AB)^* = 0$, so $B^* \subseteq \operatorname{Ann} A = B$; by symmetry, $B = B^*$.

(ii) $(J \cap \operatorname{Ann} J)^2 \subseteq J \operatorname{Ann} J = 0$, implying $J \cap \operatorname{Ann} J = 0$. But $\operatorname{Ann} J$ is an ideal of $(R, *)$, so $\operatorname{Ann} J = 0$, implying J is essential in R.

(iii) J^* is clearly essential in R, so JJ^* is essential in R; thus JJ^* is certainly essential in $(R, *)$. Q.E.D.

Theorem 8. *Let* $\hat{C} = \mathrm{cent}(R, *)$. $(Q(R), *)$ *can be characterized as follows*:

(i) *There is an injection* $(R, *) \to (Q(R), *)$ *sending* $\hat{C}$ *into* $\mathrm{cent}(Q(R), *)$.

(ii) *For any essential ideal* E *of* $\hat{C}$ *and for any* f *in* $\mathrm{Hom}_{\hat{C}}(E, R)$, *there exists* q *in* $Q(R)$ *such that* $xq = f(x)$, *all* x *in* E.

(iii) *Given* q *in* $Q(R)$, *one can find an essential ideal* E *of* $\hat{C}$ *such that* $Eq \subseteq R$.

(iv) $q = 0$ *if and only if there exists an essential ideal* E *of* $\hat{C}$ *such that* $Eq = 0$.

Proof. This is straightforward from Theorems 1 and 7, when it is noted that $\hat{C} = \mathrm{cent}(C, *)$; hence any essential ideal of $\hat{C}$ is an essential ideal of C by Lemma 3, and if E is an essential ideal of C then EE^* is an essential ideal of $\hat{C}$.

Conversely, we wish to show that for any algebra $(Q, *)$ satisfying (i)′–(iv)′, Q is the maximal quotient algebra of R. To see this, we shall show Q satisfies properties (i)–(iv) of Theorem 1. Observe that, by Lemma 3, any essential ideal in $\hat{C}$ is essential in C.

Hence (iii) and (iv) are immediate. To obtain (i), we need only show $C \subseteq \mathrm{cent}\, Q$. Indeed, given $c \in C$ and q in Q, choose an essential ideal E of $\hat{C}$ such that $Eq \subseteq R$. Then $E(cq - qc) = 0$, so $cq - qc = 0$ for all q in Q, implying $c \in \mathrm{cent}\, Q$.

Finally we need to prove (ii). Suppose E is an essential ideal of C and $f \in \mathrm{Hom}_C(E, R)$. Then E^*E is essential in $\hat{C}$ and f restricts to a $\hat{C}$-homomorphism from E^*E to R; hence there is q in Q such that $f(x) = xq$, all x in E^*E. Thus, for all x in E, $E^*(f(x) - xq) = 0$, implying $f(x) - xq = 0$, all x in E by (iv). Q.E.D.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637